Chapter A: Convex Analysis

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Line segment

Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

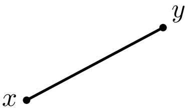
$$\theta x + (1 - \theta) y$$

is a point in between x and y if $\theta \in [0, 1]$.

The set of all points between a given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$

$$\{\theta x + (1 - \theta) y \,|\, \theta \in [0, 1]\}$$

is called the *line segment* between x and y



Convex combinations

Given $x_1, \ldots, x_k \in \mathbb{R}^n$,

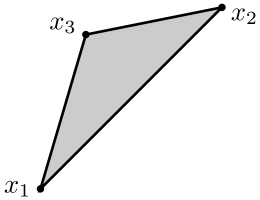
$$\theta_1 x_1 + \cdots + \theta_k x_k$$

is called a *convex combination* or a *weighted average* of $x_1, \ldots, x_k$ if $\theta_1, \ldots, \theta_k \geq 0$ and $\theta_1 + \cdots + \theta_k = 1$.

Given $x_1, \ldots, x_k \in \mathbb{R}^n$, the set of all convex combinations

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid \theta_1, \ldots, \theta_k \geq 0,\ \theta_1 + \cdots + \theta_k = 1\}$$

is called the *convex hull* of $x_1, \ldots, x_k$.



Convex sets

We say a set $C \subseteq \mathbb{R}^n$ is *convex* if

$$\theta x + (1-\theta) y \in C, \qquad \forall\, x, y \in C,\, \theta \in (0,1).$$

In other words, C is convex if $x, y \in C$ implies the line segment connecting x and y is wholly contained in C.

TODO: Add picture

Theorem.

A convex set is closed under convex combinations, i.e. if $x_1, \ldots, x_k \in C$ *for a convex set* $C \subseteq \mathbb{R}^n$*, then* $\theta_1 x_1 + \cdots + \theta_k x_k \in C$ *for any* $\theta_1, \ldots, \theta_k \geq 0$ *and* $\theta_1 + \cdots + \theta_k = 1$*.*

Proof.

□

Convex functions

We say a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y), \qquad \forall\, x, y \in \mathbb{R}^n,\ \theta \in [0,1].$$

In other words, f is convex if the line segment connecting $(x, f(x))$ and $(y, f(y))$ lies above the graph of f.

TODO: Picture

No bad local minima for cvx. functions

Theorem.
Let f be convex. Then any local minimizer is a global minimizer.

Thus, when we minimize convex functions, we never get stuck at bad local minima because there aren't any bad local minima.

Proof. TODO: Picture □

No bad local minima for cvx. functions

Theorem.
Let f be convex. Then any local minimizer is a global minimizer.

Proof. Let $x_\star \in \mathbb{R}^n$ be a local minimizer of f. Assume for contradiction that there is $y_\star \in \mathbb{R}^n$ such that $f(y_\star) < f(x_\star)$, i.e., assume for contradiction that $x_\star$ is not a global minimizer. By convexity,

$$f(\theta x_\star + (1-\theta) y_\star) \le \theta f(x_\star) + (1-\theta) f(y_\star) < f(x_\star)$$

for any $\theta \in (0,1)$, even for θ very close to 1. However, $x_\star$ is a local minimizer, so $f(\theta x_\star + (1-\theta) y_\star) \ge f(x_\star)$ for θ sufficiently close to 1, and we have a contradiction. Thus we conclude that such $y_\star$ cannot exist, i.e., $x_\star$ is a global minimizer. $\square$

Gradient provides global lower bound for cvx. functions

Theorem.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex. Assume f is differentiable at x. Then,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \qquad \forall\, y \in \mathbb{R}^n.$$

I.e., the first-order Taylor expansion of f is a global lower bound of f.

Proof. XXX
Proof by picture XXX

Gradient provides global lower bound for cvx. functions

Theorem.
Let $f: \mathbb{R}^n \to \mathbb{R}$ *be convex. Assume* f *is differentiable at* x*. Then,*

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \qquad \forall\, y \in \mathbb{R}^n.$$

Proof. By convexity,

$$f(x + \theta(y - x)) \leq (1 - \theta) f(x) + \theta f(y), \qquad \forall\, \theta \in (0, 1).$$

Reorganizing, we get

$$f(y) \geq f(x) + \frac{f(x + \theta(y - x)) - f(x)}{\theta}, \qquad \forall\, \theta \in (0, 1).$$

By taking $\theta \to 0$, we get the desired inequality. $\square$