Optimization, MATH 164
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Homework 2
Due on Friday, February 7, 2025.

Problem 1: *Sum of smooth functions is smooth.*

(a) Let f_1 be L_1-smooth and f_2 be L_2-smooth. Show that $f_1 + f_2$ is $(L_1 + L_2)$-smooth.

(b) Let f_1 be L-smooth and f_2 be affine. Show that $f_1 + f_2$ is L-smooth.

Problem 2: *Overestimating the smoothness constant.* Let $0 < L_1 < L_2$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex. Imagine the circumstance where you know f is L_2-smooth, and you know the numerical value of L_2, but, unbeknownst to you, f is furthermore L_1-smooth. (So you have overestimated the smoothness constant of f.)

(a) Show that if f is L_1-smooth, then it is L_2-smooth.

(b) Show that if f is L_1-smooth and you use

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$

with $\alpha \in (0, 1/L_1]$, then GD converges with the rate

$$f(x_k) - f(x_\star) \le \frac{1}{2\alpha k}\|x_0 - x_\star\|^2.$$

(Note that the constant is optimized at $\alpha = 1/L_1$.)

Hint. For part (b), there is no need to carry out the convergence analysis from scratch. You can use the convergence result proved in class.

Remark. The point of this problem is that overestimating the smoothness constant will lead to a worse constant in the convergence guarantee but will otherwise not break the convergence guarantee.

Problem 3: *Underestimating the smoothness constant.* Let $0 < L_1 < L_2/2$ (note the factor $1/2$) and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex. Imagine the circumstance where you think f is L_1-smooth f, but f is in fact only L_2-smooth and not L_1-smooth. (So you have underestimated the smoothness constant of f.)

(a) Show that

$$f(x) = \frac{L_2}{2}\|x\|^2$$

is (i) convex, (ii) L_2-smooth, (iii) not L_1-smooth, and (iv) has the global minimizer $x_\star = 0$.

(b) Show that GD with stepsize $\alpha = 1/L_1$ diverges unless the iteration starts at the solution.

Remark. The point of this problem is that underestimating the smoothness constant can break the convergence guarantee.

Problem 4: Show that a convex set is closed under convex combinations, i.e. if $x_1, \ldots, x_k \in C$ for a convex set $C \subseteq \mathbb{R}^n$, then $\theta_1 x_1 + \cdots + \theta_k x_k \in C$ for any $\theta_1, \ldots, \theta_k \geq 0$ and $\theta_1 + \cdots + \theta_k = 1$.

Hint. Use induction on k.

Problem 5: Show that the intersection of convex sets is convex.

Problem 6: Show that a nonnegative combination of convex functions is convex.

Problem 7: Show that a sublevel set of a convex function is convex.

Problem 8: *Stationary points of convex functions are global minimizers.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable at $x \in \mathbb{R}^n$. Show that if $\nabla f(x) = 0$, then x is a global minimizer of f.

Clarification. In class, we showed that if x is a local minimizer, then x is a global minimizer. You are being asked to show a slightly stronger result.

Problem 9: *Method of Lagrange multipliers for multiple constraints.* Consider the optimization problem with multiple equality constraints

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x), \\ \text{subject to} \quad & g_1(x) = 0 \\ & \quad \vdots \\ & g_m(x) = 0, \end{aligned}$$

where all functions are continuously differentiable. It can be shown that all extremum point $\vec{x} = (x, y, z)$ must satisfy the system of equations

$$\begin{aligned} \nabla f(\vec{x}) &= \sum_{i=1}^{m} \lambda_i \nabla g_i(\vec{x}) \\ g_i(\vec{x}) &= 0 \qquad \text{for } i = 1, \ldots, m \end{aligned}$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Use this version of the method of Lagrange multipliers to find the solution of the optimization problem

$$\begin{aligned} \underset{x,y,z \in \mathbb{R}}{\text{minimize}} \quad & 3x - y - 3z \\ \text{subject to} \quad & x + y - z = 0 \\ & x^2 + 2z^2 = 1. \end{aligned}$$

Problem 10: *Convex optimization problems have convex solution sets.* Consider the constrained optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0 \end{aligned}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ are convex. Show that the set of solutions is convex.