Optimization, MATH 164
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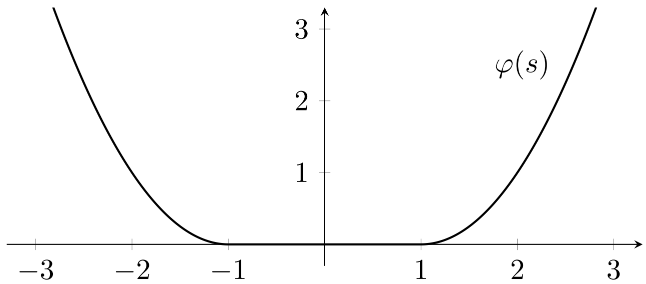
Homework 4
Due on Friday, Match 7, 2025.

Problem 1: *When AM fails to converge.* Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as

$$\varphi(s) = \begin{cases} (s+1)^2 & \text{for } s < -1 \\ 0 & \text{for } -1 \le s \le +1 \\ (s-1)^2 & \text{for } +1 < s. \end{cases}$$



Consider the optimization problem

$$\underset{x,y,z\in\mathbb{R}}{\text{minimize}} \quad \underbrace{-xy - yz - zx + \varphi(x) + \varphi(y) + \varphi(z)}_{=f(x,y,z)}.$$

Show that alternating minimization, updating x, y, and z sequentially, with starting point

$$(x^0, y^0, z^0) = (-1-\varepsilon, 1+\tfrac{1}{2}\varepsilon, -1-\tfrac{1}{4}\varepsilon)$$

with any $\varepsilon > 0$, does not converge and cycles around the points

$$(1,1,-1),\ (1,-1,-1),\ (1,-1,1),\ (-1,-1,1),\ (-1,1,1),\ (-1,1,-1).$$

Problem 2: *AM usually does not fail to converge.* Consider the setup of Problem 1. Implement alternating minimization, updating x, y, and z sequentially. Plot the trajectories in 3D and plot the function value $f(x^k, y^k, z^k)$ against the iteration count. (I encourage you to ask ChatGPT to help you write the plotting code.)

(a) Show that the optimal value of this problem is $p_\star = -\infty$.

(b) Execute coordinate minimization with starting point $(x^0, y^0, z^0) = (-1-\varepsilon, 1+\frac{1}{2}\varepsilon, -1-\frac{1}{4}\varepsilon)$ with $\varepsilon = 0.1$. Numerically show that for the first 10 iterations ($3 \times 10 = 30$ coordinate updates), the iterates behave according to the theoretical analysis of Problem 1.

(c) Execute 30 or more iterations in the setting of (b). Numerically show that the updates deviate from the theoretical analysis of Problem 1 and exhibit the behavior $f(x^k, y^k, z^k) \to -\infty$.

(d) Execute coordinate minimization with several random starting points. Numerically show that the iterations exhibit the behavior $f(x^k, y^k, z^k) \to -\infty$.

Remark. The behavior of (c) is due to numerical errors. The numerical results of this problem imply that the failure mode of AM, where the iterates cycle and fail to converge, may be exceedingly rare in practice.

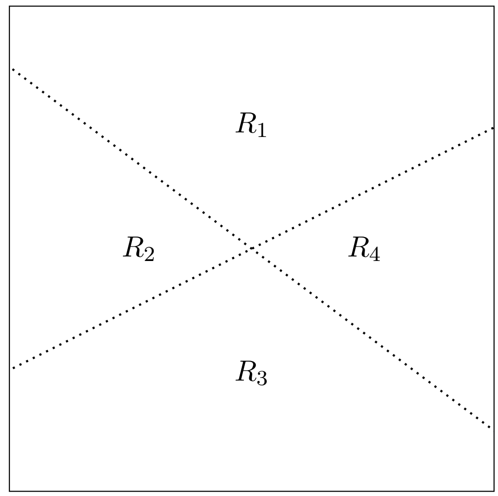
Problem 3: *When alternating minimization fails.* Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x, y) = |3x + 4y| + |x - 2y|.$$

Show that alternating minimization with "most" initializations gets stuck at a point that is not a minimizer.

Hint. Consider the following four regions and characterize the behavior in each case:

$$\begin{aligned}
R_1 &= \{(x, y) \in \mathbb{R}^2 \mid 3x + 4y > 0,\ x - 2y < 0\} \\
R_2 &= \{(x, y) \in \mathbb{R}^2 \mid 3x + 4y < 0,\ x - 2y < 0\} \\
R_3 &= \{(x, y) \in \mathbb{R}^2 \mid 3x + 4y < 0,\ x - 2y > 0\} \\
R_4 &= \{(x, y) \in \mathbb{R}^2 \mid 3x + 4y > 0,\ x - 2y > 0\}
\end{aligned}$$

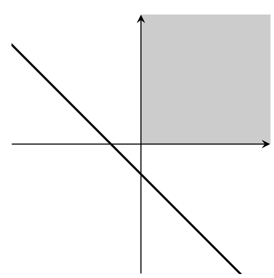


Problem 4: *Converting LPs to standard form.* Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Convert the following LP into standard form

$$\begin{array}{ll}
\underset{x \in \mathbb{R}^n,\, t \in \mathbb{R}}{\text{minimize}} & t \\
\text{subject to} & -t\mathbf{1} \leq Ax - b \leq t\mathbf{1}.
\end{array}$$

Problem 5: *Example with Farkas.* Consider the system

$$\left[\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1, \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \right]$$



(a) Appeal to Farkas' lemma to show that the fact that $z = -1$ is a feasible point for

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} z \leq 0, \quad (-1)z > 0$$

implies there is no feasible (x, y).

(b) Now, let us not directly appeal to Farkas' lemma. Assume for contradiction that (x, y) is feasible. Then, multiplying by z, we get

$$z \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (-1)z$$

Draw a contradiction and conclude that a feasible (x, y) cannot exist.