Optimization, MATH 164 E. K. Ryu Winter 2025

UCLA

Homework 5 Due on Friday, Match 14, 2025.

Problem 1: Variant of Farkas' lemma. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that

• there exists an $x \in \mathbb{R}^n$ such that $Ax \leq b$

XOR

• there exists a $y \in \mathbb{R}^m$ such that $y \ge 0$, $A^{\intercal}y = 0$, and $b^{\intercal}y < 0$.

Hint. Start by showing that

$$\left\{ \exists x \in \mathbb{R}^n : Ax \le b \right\} \quad \Leftrightarrow \quad \left\{ \exists x_+, x_- \in \mathbb{R}^n, s \in \mathbb{R}^m : \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ s \end{bmatrix} = b, \begin{bmatrix} x_+ \\ x_- \\ s \end{bmatrix} \ge 0 \right\}.$$

Problem 2: Strong duality with $-\infty = d_{\star} = p_{\star}$. Consider

 $\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^{\mathsf{T}}x \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array} \qquad \xrightarrow{\text{dual}} & \underset{y \in \mathbb{R}^m}{\text{maximize}} & b^{\mathsf{T}}y \\ \text{subject to} & A^{\mathsf{T}}y \le c. \end{array}$

- (a) Show that the dual problem is infeasible if and only if there exists a $v \in \mathbb{R}^n$ such that Av = 0, $v \ge 0$, and $c^{\intercal}v < 0$.
- (b) Assume the primal problem is feasible and the dual problem is infeasible (so $-\infty = d_{\star} \leq p_{\star} < \infty$). Show that $d_{\star} = p_{\star}$.

Problem 3: Visualizing a primal LP. Consider the

$$\begin{array}{ll} \underset{x \in \mathbb{R}^2}{\text{minimize}} & c^{\intercal} x\\ \text{subject to} & [1,-1] \, x \leq 1\\ & [1,-1] \, x \geq -2\\ & x > 0. \end{array}$$

with

$$c = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Characterize the solution set as a function of $\theta \in [0, 2\pi)$.

Problem 4: Feasible regions of an LP. Consider the shape



which is a regular pentagon with the bottom edge sitting on the x-axis, starting at x = -1 and ending at x = +1. Find an $A \in \mathbb{R}^{5\times 2}$ and $b \in \mathbb{R}^5$ such that the region defined by $Ax \leq b$ is the pentagon. For your information, the coordinates of the five vertices are

- A = (-1, 0)
- B = (1, 0)
- $C = B + 2(\cos(72^\circ), \sin(72^\circ)) \approx (1.61803, 1.90211)$
- $D = C + 2(\cos(144^\circ), \sin(144^\circ)) \approx (0, 3.07768)$
- $E = D + 2(\cos(216^\circ), \sin(216^\circ)) \approx (-1.61803, 1.90211)$