Optimization, MATH 164
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Homework 5
Due on Friday, Match 14, 2025.

Problem 1: *Variant of Farkas' lemma.* Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that

- there exists an $x \in \mathbb{R}^n$ such that $Ax \le b$

XOR

- there exists a $y \in \mathbb{R}^m$ such that $y \ge 0$, $A^\intercal y = 0$, and $b^\intercal y < 0$.

Hint. Start by showing that

$$\left\{\exists x \in \mathbb{R}^n \,:\, Ax \le b\right\} \quad \Leftrightarrow \quad \left\{\exists x_+, x_- \in \mathbb{R}^n,\, s \in \mathbb{R}^m \,:\, \begin{bmatrix} A & -A & I \end{bmatrix}\begin{bmatrix} x_+ \\ x_- \\ s \end{bmatrix} = b,\ \begin{bmatrix} x_+ \\ x_- \\ s \end{bmatrix} \ge 0\right\}.$$

Problem 2: *Strong duality with $-\infty = d_\star = p_\star$.* Consider

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & c^\intercal x \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{array}{ll} \underset{y\in\mathbb{R}^m}{\text{maximize}} & b^\intercal y \\ \text{subject to} & A^\intercal y \le c. \end{array}$$

(a) Show that the dual problem is infeasible if and only if there exists a $v \in \mathbb{R}^n$ such that $Av = 0$, $v \ge 0$, and $c^\intercal v < 0$.

(b) Assume the primal problem is feasible and the dual problem is infeasible (so $-\infty = d_\star \le p_\star < \infty$). Show that $d_\star = p_\star$.

Problem 3: *Visualizing a primal LP.* Consider the

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{minimize}} & c^\intercal x \\ \text{subject to} & [1,-1]\, x \le 1 \\ & [1,-1]\, x \ge -2 \\ & x \ge 0. \end{array}$$

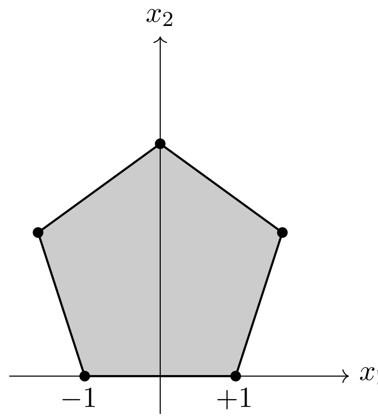
with

$$c = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Characterize the solution set as a function of $\theta \in [0, 2\pi)$.

Problem 4: *Feasible regions of an LP.* Consider the shape

$$\{x \in \mathbb{R}^2 \mid Ax \leq b\} =$$



which is a regular hexagon with the bottom edge sitting on the x-axis, starting at $x = -1$ and ending at $x = +1$. Find an $A \in \mathbb{R}^{5\times 2}$ and $b \in \mathbb{R}^2$ such that the region defined by $Ax \leq b$ is the pentagon. For your information, the coordinates of the five vertices are

- $A = (-1, 0)$
- $B = (1, 0)$
- $C = B + 2(\cos(72°), \sin(72°)) \approx (1.61803, 1.90211)$
- $D = C + 2(\cos(144°), \sin(144°)) \approx (0, 3.07768)$
- $E = D + 2(\cos(216°), \sin(216°)) \approx (-1.61803, 1.90211)$