Advanced Numerical Analysis, MATH 269A
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Homework 1
Due on Friday, October 11, 2024.

Problem 1: *Definition of a solution.* Consider the ODE

$$y' = f(t, y), \qquad y(0) = y_0$$

for $t \in [0, T]$. One says $y\colon [0, T] \to \mathbb{R}$ is a "solution" if y satisfies the equations above. Strictly speaking, however, $y'(0)$ and $y'(T)$ are not defined (at least the two-sided derivatives are not). Let's consider several definitions of a solution:

(A) y satisfies $y(0) = y_0$, $y' = f$ for $t \in [0, T]$, where y' is interpreted as one-sided derivatives at $t = 0$ and $t = T$.

(B) y satisfies $y(0) = y_0$, $y' = f$ for $t \in (0, T)$, and y is continuous at $t = 0$ and $t = T$.

(C) y satisfies

$$y(t) = y(0) + \int_0^t f(s, y(s))\, ds$$

for all $t \in [0, T]$.

Clearly, a solution of type (A) is a solution of type (B).

(a) Show that the ODE

$$y' = 1/\sqrt{y}, \qquad y(0) = 0$$

has no solution of type (A) but admits a solution of type (B).

(b) Show that a solution of type (B) is a solution of type (C).

(c) Show that the ODE with

$$f(t, y) = \begin{cases} 0 & \text{if } t \in [0, T/2) \\ 1 & \text{if } t \in [T/2, T] \end{cases}$$

admits a solution of type (C) but not a solution of type (B).

Clarification. Strictly speaking, (C) would additionally require that $f(t, y(t))$ is Legesgue integrable on $t \in [0, T]$. Then, we can measure theoretically state (C) as $y\colon [0, T] \to \mathbb{R}$ is an absolutely continuous function satisfying $y(0) = y_0$ and $y'(t) = f(t, y(t))$ for Lebesgue almost all $t \in [0, T]$.

Clarification. We are being mathematically pedantic in this problem, and $h(t) = \max\{0, t - T/2\}$ is not differentiable at $T/2$.

Problem 2: *Picard–Lindelöf theorem.* Consider the ODE of Problem 1. Let $K > 0$ and $L > 0$. Further assume that $f\colon [0,T] \times \mathbb{R} \times \mathbb{R}$ is continuous, that $|f(t,y)| \le K$ for all $(t,y) \in [0,T] \times \mathbb{R}$, and that f is L-Lipschitz in y, i.e.,

$$|f(t,y) - f(t,\tilde{y})| \le L|y - \tilde{y}|, \qquad \forall\, t \in [0,T],\ y, \tilde{y} \in \mathbb{R}.$$

For $n = 0, 1, 2, \ldots$, let $y^{(n)}\colon [0,T] \to \mathbb{R}$ be a sequence of functions. Specifically, let $y^{(0)}(t) = y_0$ for $t \in [0,T]$ and for $n = 1, \ldots, \infty$, let

$$y^{(n)}(t) = y_0 + \int_0^t f(s, y^{(n-1)}(s))\, ds, \qquad \text{for } t \in [0,T].$$

(a) Using induction, show

$$|y^{(n)}(t) - y^{(n-1)}(t)| \le \frac{K}{L}\frac{(Lt)^n}{n!}, \qquad \text{for } t \in [0,T]$$

for $n = 1, 2, \ldots$.

(b) Show that $\{y^{(n)}(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for each $t \in [0,T]$.

(c) Define

$$y^{(\infty)}(t) = \lim_{n\to\infty} y^{(n)}(t)$$

for each $t \in [0,T]$. Show that

$$\sum_{m=1}^{\infty} |y^{(m)}(t) - y^{(m-1)}(t)| \le \frac{K}{L}(e^{LT} - 1)$$

for all $t \in [0,T]$ and that

$$|y^{(\infty)}(t) - y^{(n)}(t)| \le \sum_{m=n}^{\infty} |y^{(m)}(t) - y^{(m-1)}(t)| \to 0$$

uniformly in t. (Since the uniform limit of a sequence of continuous functions is continuous, i.e., since $\mathcal{C}([0,T], \|\cdot\|_\infty)$ is a Banach space, $y^{(\infty)}$ is a continuous function.)

(d) In the following steps, justify $(*)$, $(\dagger)$, and $(\#)$:

$$\begin{aligned} y^{(\infty)}(t) &= \lim_{n\to\infty} y^{(n)}(t) \\ &\overset{(*)}{=} y_0 + \lim_{n\to\infty} \int_0^t f(s, y^{(n-1)}(s))\, ds \\ &\overset{(\dagger)}{=} y_0 + \int_0^t \lim_{n\to\infty} f(s, y^{(n-1)}(s))\, ds \\ &\overset{(\#)}{=} y_0 + \int_0^t f(s, y^{(\infty)}(s))\, ds. \end{aligned}$$

(e) Show that $y^{(\infty)}$ is a solution to the ODE of type (A) in the sense of Problem 1.

Remark. Thus, existence is established.

Problem 3: *Grönwall's inequality I.* Let $L > 0$. Let $\mathcal{E}(t) \geq 0$ be a differentiable function satisfying

$$\frac{d}{dt}\mathcal{E}(t) \leq L\mathcal{E}(t), \qquad \text{for } t > 0.$$

Show that

$$\mathcal{E}(t) \leq e^{Lt}\mathcal{E}(0), \qquad \text{for } t \geq 0.$$

Hint. Show that $e^{-Lt}\mathcal{E}(t)$ is non-increasing.

Remark. The differentiability requirement can be relaxed to absolute continuity.

Problem 4: *Stability and uniqueness of solution.* Let $y\colon [0,T] \to \mathbb{R}$ and $\hat{y}\colon [0,T] \to \mathbb{R}$ be continuously differentiable functions satisfying $y' = f(t,y)$ and $\hat{y}' = f(t,\hat{y})$ for $t \in [0,T]$, but with different initial conditions $y(0) = y_0$ and $\hat{y}(0) = \hat{y}_0$. Assume $f(t,y)$ is L-Lipschitz in y. Show that

$$|y(t) - \hat{y}(t)| \leq e^{Lt}|y(0) - \hat{y}(0)|, \qquad \text{for } t \in [0,T].$$

Remark. By plugging in $\hat{y}_0 = y_0$, uniqueness is established.

Remark. This result is considered to be the second part of the Picard–Lindelöf theorem, while the result of Problem 2 is considered the first part.

Problem 5: *Stability of Euler.* Consider the ODE

$$y' = -100(y - \sin(t)), \qquad y(0) = 1$$

for $t \in [0, 3]$ and consider the following implementation of Euler's method.

```
import math as math
import matplotlib.pyplot as plt

# Sepcify ODE
f = lambda t,y: -100*(y-math.sin(t))
y0 = 1.
T = 3.0

# Set stepsize
N = 1000
h = T/N

# Pre-allocate list for timesteps and solution
t_list = [h*i for i in range(N+1)]
y_list = [0 for i in range(N+1)]
y_list[0] = y0 # set initial condition

# Euler's method
for ind in range(N):
  t, y = t_list[ind], y_list[ind]
  y_list[ind + 1] = y + h * f(t, y)

# Plotting code
plt.plot(t_list, y_list, label="Euler's Method")
plt.xlabel('t')
plt.ylabel('y')
plt.title('Euler Method Solution to ODE')
plt.legend()
plt.grid(True)
plt.savefig('plot.png')
plt.show()
```

(a) At around what value of N does the numerical solution "stabilize"? What value of h does this correspond to?

(b) For simplicity, let us ignore the $\sin(t)$ term and consider the ODE $y' = -100y$ instead. (For small $t > 0$, the $\sin(t)$ term is indeed small compared to the initial condition $y(0) = 1$.) Then, the Euler method corresponds to

$$y_{n+1} = (1 - h100)y_n$$

For what values of $h > 0$ do we have $y_n \to 0$?

Problem 6: *Global solution may expectedly fail to exist.* Consider the ODE

$$y' = y\tan(t), \qquad y(0) = 1$$

for $t \geq 0$, which has the solution $y(t) = 1/\cos(t)$. Note, this solution is only valid $t < \pi/2$.

(a) Why can we not apply the Picard–Lindelöf theorem to get a solution on an aibitraty time interval $[0, T]$? What conditions are violated?

(b) Implement Euler's method to simulate the local solution. Do you observe the blowup?

Remark. We say that a global solution (a solution for all of $t \geq 0$) does not exist for this problem. This is expected since the RHS blows up at finite time. When an ODE contains terms with singularities, pay close attention to their effects in the simulations.

Problem 7: *Global solution may unexpectedly fail to exist.* Consider the ODE

$$y' = y^2, \qquad y(0) = 1$$

for $t \geq 0$, which has the solution $y(t) = \frac{1}{1-t}$. Note, this solution is only valid $t < 1$. Implement Euler's method to simulate the local solution. Do you observe the blowup?

Remark. For this ODE, the blow-up at finite time is perhaps less expected, since the RHS is smooth and seemingly well behaved.

Problem 8: *Singular ODE.* Consider the ODE

$$y' = -\frac{y}{\sqrt{t}}, \qquad y(0) = 1$$

for $t \geq 0$, which has the solution $y(t) = \exp(-2\sqrt{t})$. Can you implement Euler's method?

Remark. A singularity in the ODE does not necessarily mean the solution blows up.

Problem 9: *Non-unique solutions.* Consider the ODE

$$y' = y^{2/3}, \qquad y(0) = 0$$

for $t \geq 0$.

(a) Show that for any $\alpha \geq 0$,

$$y(t) = \begin{cases} 0 & \text{for } t < \alpha \\ \frac{1}{27}(t-\alpha)^3 & \text{for } \alpha \leq t \end{cases}$$

is a solution. (So, the solution is not unique.)

(b) Why can't we apply the Picard–Lindelöf theorem (Problem 4) to establish uniqueness? What conditions are violated?

(c) Implement Euler's method. Which solution do you get?