Advanced Numerical Analysis, MATH 269A
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Homework 2
Due on Monday, October 21, 2024.

Problem 1: *Matrix exponential.* In this problem, we quickly review the matrix exponential.

(a) Consider the complex scalar ODE

$$\dot{y} = \lambda y, \qquad y(0) = y_0$$

for $t \geq 0$, where $\lambda \in \mathbb{C}$. (This is called the *test equation.*) Assume $y_0 \neq 0$. Show that $\lim_{t\to\infty} y(t) = 0$ if and only if $\mathrm{Re}(\lambda) < 0$.

(b) For $A \in \mathbb{R}^{d\times d}$, the matrix exponential is defined as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

where $A^0 = I$ by convention. Consider the ODE

$$\dot{y} = Ay, \qquad y(0) = y_0$$

for $t \geq 0$, where $y(t) \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$. Show that

$$y(t) = e^{tA} y_0$$

is a solution.

(c) Let $A \in \mathbb{R}^{d\times d}$ is diagonalizable, and write

$$A = V \mathrm{diag}(\lambda_1, \ldots, \lambda_d) V^{-1}.$$

Show that

$$e^{tA} = V \mathrm{diag}(e^{t\lambda_1}, \ldots, e^{t\lambda_d}) V^{-1}.$$

(d) Consider the ODE

$$\dot{y} = Ay, \qquad y(0) = y_0$$

for $t \geq 0$. Assume $A \in \mathbb{R}^{d\times d}$ is diagonalizable, i.e., $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix. Show that [$\lim_{t\to\infty} y(t) = 0$ for any $y_0 \in \mathbb{R}^d$] if and only if $\mathrm{Re}(\lambda_1), \ldots, \mathrm{Re}(\lambda_d) < 0$.

Remark. Solutions to $\dot{y} = \lambda y$ and $\dot{y} = Ay$ uniquely exists. Also, e^A is well-defined, i.e., the power series always converges. You may use these facts without proof.

Remark. You may argue without proof that

$$\frac{d}{dt} \sum_{k=0}^{\infty} = \sum_{k=0}^{\infty} \frac{d}{dt}.$$

Remark. The conclusion of (d) holds without the assumption of diagonalizability, and the proof can be done using the Jordan canonical form.

Problem 2: *Stepsize control (impractical version).* Consider the ODE

$$y' = f(t, y), \qquad y(0) = y_0$$

for $t \in [0, T]$ with a solution $y(\cdot)$. Let $L > 0$, and assume $f(t, y)$ is L-Lipschitz in y. Consider the forward Euler method with non-uniform steps

$$\begin{aligned} t_{n+1} &= t_n + h_n \\ y_{n+1} &= y_n + h_n f(t_n, y_n) \end{aligned}$$

with $h_n > 0$ for $n = 0, 1, \ldots, N-1$, $\tau_0 = 0$, and $t_N = T$. Define the local truncation error τ_n via

$$y(t_{n+1}) = y(t_n) + h_n f(t_n, y(t_n)) + \tau_n, \qquad \text{for } n = 0, \ldots, N-1.$$

(a) Show that

$$\begin{aligned} |y(t_N) - y_N| &\leq \sum_{j=0}^{n-1} \frac{|\tau_j|}{h_j} e^{(t_n - t_{j+1})L} h_j \\ &\leq \left(\max_{j=0,\ldots,N-1} \frac{|\tau_j|}{h_j} \right) \sum_{j=0}^{N-1} e^{(t_N - t_{j+1})L} h_j \\ &\leq \max_{j=0,\ldots,N-1} \frac{|\tau_j|}{h_j} e^{TL} \int_0^T e^{-tL}\, dt \\ &= \max_{j=0,\ldots,N-1} \frac{|\tau_j|}{h_j} \frac{1}{L}(e^{TL} - 1). \end{aligned}$$

(b) Assume $f(t, y)$ is continuously differentiable. Let $h_1, \ldots, h_{n-1}$ be given and fixed. Show that $\tau_n = o(h_n)$ as $h_n \to 0$, i.e., show that

$$\limsup_{h_n \to \infty} |\tau_n| / h_n = 0.$$

(c) Let $\varepsilon > 0$. Assume we can somehow choose each $h_0, \ldots, h_{N-1}$ such that $|\tau_n|/h_n \leq \varepsilon$ for all $n = 0, \ldots, N-1$. Then, show that

$$|y(t_N) - y_N| \leq \frac{\varepsilon}{L}(e^{TL} - 1).$$

Remark. So, given any $h_1, \ldots, h_{n-1}$, we can make $|\tau_n|/h_n$ as small as we want by making h_n sufficiently small. In theory, this stepsize control scheme would allow us to achieve arbitrarily small global error. (Although we do not yet know how large N will be or if $N = \infty$.) What is missing, however, is an implementable mechanism to determine/approximate the magnitude of $|\tau_n|/h_n$ in choosing h_n.

Problem 3: *Error analysis of Heun.* Consider the ODE

$$y' = f(t,y), \qquad y(0) = y_0$$

for $t \in [0,T]$. Assume all functions are sufficiently smooth. Let $L > 0$ and assume $f(t,y)$ is L-Lipschitz in y. Consider the Heun's method

$$\begin{aligned} y^{n+1/2} &= y_n + hf(t_n, y_n) \\ y^{n+1} &= y_n + \frac{h}{2}\big(f(t_n,y_n) + f(t_{n+1}, y_{n+1/2})\big), \end{aligned}$$

also written as

$$y^{n+1} = y_n + \frac{h}{2}\big(f(t_n,y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))\big)$$

for $n = 0,1,\ldots,N-1$. Define the local truncation error (LTE) τ_n as

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\big(f(t_n,y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n,y(t_n)))\big) + \tau_n$$

for $n = 0,1,\ldots,N-1$.

(a) Show that

$$\tau_n = \frac{h^3}{6}\Big(f_{tt}+2f_{ty}f+f_{yy}f^2+f_yf_t+(f_y)^2f\Big)\Big|_{\substack{t=t_n+h_1^*\\ y=y(t_n+h_1^*)}} - \frac{h^3}{4}\Big(f_{tt}+2f_{ty}f+f_{yy}f^2\Big)\Big|_{\substack{t=t_n+h_2^*\\ y=y(t_n)+h_2^*f(t_n,y(t_n))}}$$

for some $h_1^*, h_2^* \in [0,h]$, where

$$f = f(t,y), \qquad f_t = \frac{\partial f(t,y)}{\partial t}, \qquad f_y = \frac{\partial f(t,y)}{\partial y}$$
$$f_{tt} = \frac{\partial^2 f(t,y)}{\partial t^2}, \qquad f_{ty} = \frac{\partial^2 f(t,y)}{\partial t \partial y}, \qquad f_{yy} = \frac{\partial^2 f(t,y)}{\partial y^2}.$$

(b) Assume $M \geq 1$ satisfies

$$\sup_{t\in[0,T],\, y\in\mathbb{R}} \max\{|f|,|f_t|,|f_y|,|f_{tt}|,|f_{ty}|,|f_{yy}|\} \leq M.$$

Show that

$$\tau_n \leq 2M^3h^3 \qquad \text{for } n = 0,\ldots,N-1.$$

(c) Still assuming the M-bound of part (b), show that

$$\max_{n=0,\ldots,N} |y(t_n) - y_n| \leq \frac{2M^3h^2}{L}(e^{TL} - 1).$$

Hint. Let

$$\tau_n = \underbrace{y(t_n+h) - y(t_n)}_{\stackrel{\text{def}}{=}\tilde{\tau}(h)} - \frac{h}{2}\underbrace{\big(f(t_n,y(t_n)) + f(t_n+h, y(t_n) + hf(t_n,y(t_n)))\big)}_{\stackrel{\text{def}}{=}\eta(h)},$$

and use the Taylor remainder theorem to get

$$\begin{aligned} \tilde{\tau}(h) &= \tau(0) + h\tau'(0) + \frac{h^2}{2}\tau''(0) + \frac{h^3}{6}\tau^{(3)}(h_1^*), && \text{for some } h_1^* \in [0,h] \\ \eta(h) &= \eta(0) + h\eta'(0) + \frac{h^2}{2}\eta''(h_2^*), && \text{for some } h_2^* \in [0,h]. \end{aligned}$$

Problem 4: *Stability of implicit Euler.* Consider the ODE

$$y' = -100(y - \sin(t)), \qquad y(0) = 1$$

for $t \in [0, 3]$. Consider the following implementation of implicit Euler using Newton's method.

```
import math as math
import matplotlib.pyplot as plt

# Sepcify ODE
f = lambda t,y: -100*(y-math.sin(t))
fp = lambda t,y: -100
y0 = 1.
T = 3.0

# Set stepsize
N = 1000
h = T/N

# Pre-allocate list for timesteps and solution
t_list = [h*i for i in range(N+1)]
y_list = [0 for i in range(N+1)]
y_list[0] = y0 # set initial condition

# Implicit Euler
for ind in range(N):
  tnxt, y = t_list[ind]+h, y_list[ind]  #t_next, y_curr
  ynxt = y  #y_next
  while True:  #Newton iteration
    update = (ynxt-y-h*f(tnxt,ynxt))/(1-h*fp(tnxt,ynxt))
    if abs(update)<1e-10: break
    ynxt -= (ynxt-y-h*f(tnxt,ynxt))/(1-h*fp(tnxt,ynxt))
  y_list[ind + 1] = ynxt

# Plotting code
plt.plot(t_list, y_list, label="Implict Euler's Method")
plt.xlabel('t')
plt.ylabel('y')
plt.title('Euler Method Solution to ODE')
plt.legend()
plt.grid(True)
plt.savefig('plot.png')
plt.show()
```

At around what value of N does the numerical solution "stabilize"? How is the behavior different from that of explicit Euler?

Problem 5: *Implicit Euler update is well defined.* Consider the equation defining the implicit Euler update:

$$y = y_n + hf(t_{n+1}, y) \in \mathbb{R}^d,$$

where $f(t_{n+1}, \cdot)$ is continuously differentiable in a neighborhood of $y = y_n$. Show that a solution y uniquely exists for sufficiently small $h > 0$.

Hint. Let

$$G(h, y) = y - y_n - hf(t_{n+1}, y).$$

Clearly, $G(0, y) = 0$ is solved with $y = y_n$. Use the implicit function theorem.

Problem 6: *Implicit Euler on a singular ODE.* Consider the ODE

$$y' = -\frac{y}{\sqrt{t}}, \qquad y(0) = 1$$

for $t \geq 0$.

(a) Is the implicit Euler update well defined for sufficiently small $h > 0$?

(b) Implement implicit Euler. For what values of N do the results look good?

Problem 7: *Non-unique solutions and implicit Euler.* Consider the ODE

$$y' = y^{2/3}, \qquad y(0) = 0$$

for $t \geq 0$. Show that the first-step of implicit Euler with $h > 0$ is not uniquely defined.