Advanced Numerical Analysis, MATH 269A
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UCLA

Homework 3
Due on Friday, October 25, 2024.
These problems are part of homework 3. More problems will be assigned.

Problem 1: *Asymptotic orders of Euler methods.* Consider the ODE

$$y' = -100(y - \sin(t)), \qquad y(0) = 1$$

for $t \in [0, 3]$. Implement explicit Euler and implicit Euler. Use the final step of the implicit Euler solution with $N = 10^7$ as a proxy for the true value of $y(T)$. Numerically observe that

$$y(T) - y_N \sim Ch^p \qquad \text{as } N \to \infty.$$

(a) What are the estimated orders p of explicit and implicit Euler? What are the estimated values of the constant C?

(b) Consider an incorrect implementation of implicit Euler with the update

$$y_{n+1} = y_n + hf(t_n, y_{n+1})$$

for $n = 0, \ldots, N-1$. What is the estimated order p? What is the estimated value of the constant C?

Problem 2: *Debugging through asymptotic order.* Consider the ODE

$$y' = 4t^2 \cos(y), \qquad y(0) = 0$$

for $t \in [0, 10]$. Recall that Heun's method has the form

$$y^{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\big) \qquad \text{for } n = 0, 1, \ldots, N-1.$$

Consider an incorrect version of Heun's method:

$$y^{n+1} = y_n + hf(t_{n+1}, y_n + hf(t_n, y_n)) \qquad \text{for } n = 0, 1, \ldots, N-1.$$

For both methods, plot

$$\log_2 \left| \frac{y^{(M/2)}(T) - y^{(M)}(T)}{y^{(M/4)}(T) - y^{(M/2)}(T)} \right|$$

as $M \to \infty$, where $y^{(m)}(T)$ denotes the output of the simulation at the final step (the m-th step) with $N = m$ total steps. How do the estimated orders of the two methods compare?

Remark. In optimization, this "incorrect" method is called the *extragradient method.*

Problem 3: *If Newton converges, the limit is a root.* Assume $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable. Consider the Newton iteration

$$x_{n+1} = x_n - \big(Df(x_n)\big)^{-1} f(x_n) \qquad \text{for } n = 0, 1, \ldots.$$

Assume $Df(x_n)$ is invertible for $n = 0, 1, \ldots$ so that the iterates are well defined. Assume $x_n \to x_\infty$ and $Df(x_\infty)$ is invertible. Show that $f(x_\infty) = 0$.