Advanced Numerical Analysis, MATH 269A
E. K. Ryu
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Homework 5
Due on Monday, December 9, 2024.

**Problem 1:** *Coefficients of Adams and BDF methods.* The $\beta_0, \dots, \beta_k$ coefficients of the $k$-step Adams–Bashforth method can be computed with the following symbolic computation.

```
import sympy as sp

k=3
beta_list = []

for j in range(k):
  s = sp.symbols('s', real=True)
  m = sp.symbols('m', integer = True)
  prod = sp.Product((k-1-m+s)/(j-m), (m, 0, j-1)).doit()
  prod *= sp.Product((k-1-m+s)/(j-m), (m, j+1, k-1)).doit()
  beta_list.append( sp.integrate(prod, (s, 0, 1)) )

print("beta_{0},...,beta_{k-1}): ", beta_list)
print("beta_{k}: ", 0)
```

(a) Write a program to compute the coefficients of the $k$-step Adams–Moulton and BDF methods.

(b) Using `numpy.roots`, show that the BDF methods are strongly stable for $k \le 6$, but unstable for $k = 7, 8, 9$.

**Problem 2:** *Linear multistep method without stability.* Consider the explicit linear 2-step multistep method

$$y_{n+2} = -4y_{n+1} + 5y_n + 4hf_{n+1} + 2hf_n \qquad \text{for } n = 0, 1, \dots.$$

(a) Show that the linear multistep method has order 3. (Note that the order is higher than the order of 2-step Adams–Bashforth or the 2-step Adams–Moulton methods.)

(b) Show that this method does not satisfy the Dahlquist root condition.

(c) Implement this method on the ODE

$$y' = -y, \qquad y(0) = 1,$$

which, of course, has the analytical solution $y(t) = e^{-t}$. Use the analytical solution to obtain the initial value for $y_1$. Observe that the linear multistep method horribly diverges.

**Problem 3:** *Operator theoretic derivation of solutions to linear difference equations.* Consider the $k$-th order homogeneous linear difference equation

$$\alpha_k y_{n+k} + \cdots + \alpha_1 y_{n+1} + \alpha_0 y_n = 0 \qquad \text{for } n = 0, 1, \ldots,$$

where $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that $\alpha_0 \neq 0$, $\alpha_k \neq 0$. Let $\mathbb{N} = \{0, 1, \ldots\}$. Define $L\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ as

$$L\colon \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} \mapsto \begin{bmatrix} \alpha_k y_k + \cdots + \alpha_1 y_1 + \alpha_0 y_0 \\ \alpha_k y_{1+k} + \cdots + \alpha_1 y_2 + \alpha_0 y_1 \\ \alpha_k y_{2+k} + \cdots + \alpha_1 y_3 + \alpha_0 y_2 \\ \vdots \end{bmatrix}.$$

Let

$$\rho(z) = \alpha_k z^k + \cdots + \alpha_1 z + \alpha_0,$$

and let $z_1, \ldots, z_\ell \in \mathbb{R}$ with $1 \leq \ell \leq k$ be the distinct roots $\rho$ respectively with multiplicities $m_1, \ldots, m_\ell$. Let $E\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the left-shift operator, i.e.,

$$E\colon \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ \vdots \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \\ \vdots \end{bmatrix}.$$

Let $I\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the identity operator. Let

$$E^0 = I, \qquad E^1 = E, \qquad E^2 = E \circ E, \qquad E^3 = E \circ E \circ E, \qquad \cdots \qquad .$$

(a) Show that $L = \rho(E)$, i.e.,

$$L = \sum_{t=0}^{k} \alpha_t E^t.$$

(b) Show that

$$L = C \prod_{r=1}^{\ell} (E - z_i I)^{m_r}$$

for some $C \neq 0$.

(c) Show that for any $z \in \mathbb{C}$,

$$(E - zI)^k \big[\{n^{k-1} z^n\}\big] = \{0\}$$

for $k = 1, 2, \ldots$.

(d) Show that

$$L\big[\{p(n) z_r^n\}\big] = 0$$

for any polynomial $p(n)$ of degree less than $m_r$ for $r = 1, \ldots, \ell$.