Optimization Theory, MATH 273A
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Homework 2
Due on Wednesday, October 29, 2025.

**Problem 1:** *Convexity and (asymmetric) positive semidefinite Hessians.* Show that A twice differentiable multivariate function $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$$\nabla^2 f(x) + \left(\nabla^2 f(x)\right)^\intercal \succeq 0, \qquad \text{for all } x \in \mathbb{R}^n.$$

**Problem 2:** *Convexity inequality is equivalent to convexity.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. Assume

$$f(y) \ge f(x) + \langle \nabla f(x), y - x\rangle, \qquad \forall\, x, y \in \mathbb{R}^n.$$

Show that $f$ is convex, i.e., show

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y), \qquad \forall\, x, y \in \mathbb{R}^n,\ \theta \in [0,1].$$

**Problem 3:** *Cocoercivity implies $L$-smoothness.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable and $L > 0$. Assume the cociercivity inequality, i.e., assume

$$f(y) \ge f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2, \qquad \forall\, x, y \in \mathbb{R}^n.$$

(a) Let $a, b \in \mathbb{R}^n$ and $\eta > 0$. Show that

$$\langle a, b\rangle = \frac{1}{2\eta}\|a\|^2 + \frac{\eta}{2}\|b\|^2 - \frac{1}{2\eta}\|a - \eta b\|^2 \le \frac{1}{2\eta}\|a\|^2 + \frac{\eta}{2}\|b\|^2$$

(b) Show that

$$f(x) \le f(y) + \langle \nabla f(y), x - y\rangle + \langle \nabla f(x) - \nabla f(y), x - y\rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2, \qquad \forall\, x, y \in \mathbb{R}^n.$$

(c) Show that

$$f(y) + \langle \nabla f(y), x - y\rangle - \frac{L}{2}\|x-y\|^2 \le f(y) + \langle \nabla f(y), x - y\rangle \le f(x) \le f(y) + \langle \nabla f(y), x - y\rangle + \frac{L}{2}\|x-y\|^2$$

for all $x, y \in \mathbb{R}^n$.

*Remark.* The inequality of part (a) is called the Peter–Paul inequality or Young's inequality.

**Problem 4:** Show that strong convexity implies strict convexity.

**Problem 5:** *Point convergence rate implies function-value rate.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be $\mu$-strongly convex and $L$-smooth. Let $x_\star = \operatorname{argmin} f$.

(a) Show that

$$\tfrac{\mu}{2}\|x - x_\star\|^2 \le f(x) - f(x_\star) \le \tfrac{L}{2}\|x - x_\star\|^2, \qquad \forall\, x \in \mathbb{R}^n.$$

(b) Assume $\{x_k\}_{k\in\mathbb{N}}$ is a sequence satisfying

$$\|x_k - x_\star\|^2 \le \big(1 - \tfrac{\mu}{L}\big)^k \|x_0 - x_\star\|^2, \qquad \text{for } k = 0, 1, \dots.$$

Show that

$$f(x_k) - f(x_\star) \le \tfrac{L}{\mu}\big(1 - \tfrac{\mu}{L}\big)^k \big(f(x_0) - f(x_\star)\big), \qquad \text{for } k = 0, 1, \dots.$$

**Problem 6:** *Point convergence rate implies gradient-norm rate.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be $\mu$-strongly convex and $L$-smooth. Let $x_\star = \operatorname{argmin} f$.

(a) Show that

$$\mu\|x - x_\star\| \le \|\nabla f(x)\| \le L\|x - x_\star\|, \qquad \forall\, x \in \mathbb{R}^n.$$

(b) Assume $\{x_k\}_{k\in\mathbb{N}}$ is a sequence satisfying

$$\|x_k - x_\star\|^2 \le \big(1 - \tfrac{\mu}{L}\big)^k \|x_0 - x_\star\|^2, \qquad \text{for } k = 0, 1, \dots.$$

Show that

$$\|\nabla f(x_k)\| \le \tfrac{L}{\mu}\big(1 - \tfrac{\mu}{L}\big)^{k/2} \|\nabla f(x_0)\|, \qquad \text{for } k = 0, 1, \dots.$$

**Problem 7:** *Epigraph of convex functions.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$. Define the *epigraph* of $f$ as

$$\operatorname{epi}(f) = \{(x, t) \mid f(x) \le t,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}\} \subset \mathbb{R}^{d+1}.$$

Show that $f$ is convex if and only if its $\operatorname{epi}(f)$ is convex.

**Problem 8:** *Strict separating hyperplane theorem.* Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set, and let $z \in \mathbb{R}^n$. Show that if $z \notin C$, then there is a $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ such that $y \neq 0$

$$\begin{aligned} y^\intercal x &< \beta, \qquad \forall\, x \in C \\ y^\intercal z &> \beta. \end{aligned}$$

*Hint.* Let $\beta = \frac{1}{2}\langle y, z\rangle + \frac{1}{2}\langle y, \Pi_C(z)\rangle$.

**Problem 9:** *First-order (necessary) optimality condition for constrained optimization.* Let $C \subset \mathbb{R}^n$ be nonempty closed convex and $f \colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. Do not assume $f$ is convex.

(a) Show that if $x_\star \in \operatorname{argmin}_{x \in C} f(x)$, then

$$\langle \nabla f(x_\star), x - x_\star\rangle \geq 0, \qquad \forall\, x \in C.$$

(b) Assume $z \in C$ satisfies

$$\langle \nabla f(z), x - z\rangle \geq 0, \qquad \forall\, x \in C.$$

Why is this not sufficient to ensure $z \in \operatorname{argmin}_{x \in C} f(x)$?