Topics in Applied Mathematics: Infinitely Large Neural Networks, 3341.751
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Homework 3
Due 5pm, Friday, April 15, 2022

Problem 1: Let $\mathcal{H}$ be an RKHS on a nonempty set $\mathcal{X}$ with RK K. Let $N \in \mathbb{N}$, $x_1, \dots, x_N \in \mathcal{X}$, and

$$\mathcal{S} = \mathrm{span}(\{K(\cdot, x_i)\}_{i=1}^N) \subseteq \mathcal{H}.$$

Show if $f \in \mathcal{S}^\perp$, then $f(x_i) = 0$ for all $i = 1, \dots, N$.

Problem 2: Let $\mathcal{X}$ be a nonempty set. Let K_1 and K_2 be strictly PDKs mapping $\mathcal{X} \times \mathcal{X}$ to $\mathbb{R}$. Show that $K_1 K_2$ is strictly a PDK.

Problem 3: *A discontinuous kernel.* Let $\mathcal{X} = \mathbb{R}$. Show that $K \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$K(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

is a positive definite kernel. Furthermore, show that

$$\mathcal{H} = \left\{ f = \sum_{i=1}^N \alpha_i \mathbf{1}_{\{x_i\}} \,\middle|\, N \in \mathbb{N} \cup \{\infty\},\ \{c_i\}_{i=1}^n \subset \mathbb{R},\ \{x_i\}_{i=1}^n \subset \mathbb{R},\ \|f\|_{\mathcal{H}} < \infty \right\}$$

with inner product

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{x \in \mathbb{R}} f(x) g(x) = \sum_{x : f(x) \neq 0,\, g(x) \neq 0} f(x) g(x), \qquad \forall\, f, g \in \mathcal{H}$$

is the corresponding RKHS.

Problem 4: Let $\mathcal{X} = \mathbb{R}$. Describe the RKHS and its inner product corresponding to the PDK $K \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$K(x, x') = \cos(x - x').$$

Hint. Define

$$\mathcal{F}[a \cos(x - b)] = \begin{bmatrix} \pi a e^{-ib} \\ \pi a e^{ib} \end{bmatrix} \in \mathbb{C}^2.$$

Since

$$\begin{aligned} a \cos(x - b) &= \frac{1}{2\pi} (\pi a e^{-ib} e^{ix} + \pi a e^{ib} e^{-ix}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\omega x} ((\mathcal{F}[a \cos(x - b)])_1 \delta_{\{-1\}}(\omega) + (\mathcal{F}[a \cos(x - b)])_2 \delta_{\{1\}}(\omega))\, d\omega, \end{aligned}$$

where δ_c is the dirac delta "function" centered at c, we can interpret $\mathcal{F}$ as something analogous to the Fourier transform.

Problem 5: *Laplace kernel.* Let $\gamma > 0$ and $\mathcal{X} = \mathbb{R}$. Let

$$\mathcal{H} = \{f : \mathbb{R} \to \mathbb{R} \,|\, \|f\|_{\mathcal{H}} < \infty\}, \qquad \langle f, g\rangle_{\mathcal{H}} = \Re \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}}(\omega)\frac{\gamma^2 + \omega^2}{\gamma}\, d\omega.$$

Show that $\mathcal{H}$ is an RKHS with RK

$$K(x, x') = \kappa(x - x'), \qquad \kappa(t) = \frac{1}{2}e^{-\gamma|t|}.$$

You may use the following fact without proof:

$$\hat{\kappa}(\omega) = \int_{\mathbb{R}} e^{i\omega t}\kappa(t)\, dt = \frac{\gamma}{\gamma^2 + \omega^2}.$$

Problem 6: Let $\mathcal{X} = \mathbb{R}^d$. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive definite kernel that is bounded and continuous as a function. For $\mu, \mu' \in \mathcal{M}(\mathbb{R}^d)$, show that

$$\int_{\mathbb{R}^d} K(x, x')\, d\mu(x'),\ \forall\, x \in \mathcal{X}, \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x, x')\, d\mu(x)d\mu'(x')$$

are well defined, i.e., show that

$$\int_{\mathbb{R}^d} |K(x, x')|\, d|\mu|(x') < \infty,\ \forall\, x \in \mathcal{X}, \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K(x, x')|\, d|\mu|(x)d|\mu'|(x') < \infty.$$

Clarification. $\mathcal{M}(\mathbb{R}^d)$ is the space of *finite* signed measures and K being bounded means $\sup_{x,x'} |K(x, x')| < \infty$.
Hint. Note that $K(x, x') = \langle K(\cdot, x), K(\cdot, x')\rangle_{\mathcal{H}}$.

Problem 7: *Span of kernel embeddings.* Let $\mathcal{X} = \mathbb{R}^d$. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive definite kernel that is bounded and continuous as a function, and let $\mathcal{H}$ be the RKHS corresponding to K. For $\mu \in \mathcal{M}(\mathbb{R}^d)$, let

$$f_\mu(x) = \int_{\mathbb{R}^d} K(x, x')\, d\mu(x')$$

and

$$\langle f_\mu, f_{\mu'}\rangle_{\mathcal{H}_1} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x, x')\, d\mu(x)d\mu'(x'), \qquad \forall\, \mu, \mu' \in \mathcal{M}(\mathbb{R}^d).$$

Also, let

$$\mathcal{H}_1 = \{f_\mu \,|\, \mu \in \mathcal{M}(\mathbb{R}^d)\}$$

where $\|f_\mu\|^2_{\mathcal{H}_1} = \langle f_\mu, f_\mu\rangle_{\mathcal{H}_1}$. Show that $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}$, where $\mathcal{H}_0$ is the pre-Hilbert space defined in the proof of the Moore–Aronszajn theorem, and that

$$\langle f_\mu, f_{\mu'}\rangle_{\mathcal{H}_1} = \langle f_\mu, f_{\mu'}\rangle_{\mathcal{H}}, \qquad \forall\, f_\mu, f_{\mu'} \in \mathcal{H}_1.$$

Remark. As we will see in subsequent problems, $\mathcal{H}_1$ may not be complete.
Remark. f_μ is referred to as the kernel mean embedding when μ is a probability measure.

Problem 8: Let $\mathcal{X} = \mathbb{R}$. Consider the PDK $K\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$K(x, x') = e^{-(x-x')^2}.$$

Let $\mathcal{H}$ be the corresponding RKHS. For $h \neq 0$, let

$$f_h(x) = \frac{1}{h}(e^{-(x-h)^2} - e^{-x^2}) \in \mathcal{H}_0.$$

Show that f_h is a Cauchy sequence in $\mathcal{H}_0$ as $h \to 0$. Show that and $f_h \to f_0$ pointwise as $h \to 0$, where

$$f_0(x) = 2xe^{-x^2}.$$

Conclude that $f_0 \in \mathcal{H}$.

Remark. In fact, $p(x)e^{-(x-c)^2}$ is in $\mathcal{H}$, for any polynomial p and $c \in \mathbb{R}$.

Problem 9: Let $\mathcal{X} = \mathbb{R}$. Let K and f_0 be as defined in Problem 8. Let $\mathcal{H}_1$ be as defined in Problem 7. Show that $f_0 \notin \mathcal{H}_1$.

Hint. Let $f_\mu \in \mathcal{H}_1$, and assume for contradiction that $f_0 = f_\mu$. Consider the Fourier transforms of f_0 and f_μ to obtain an explicit form of $\hat{\mu}$, the Fourier transform of μ. Draw a contradiction against the assumption that μ is a finite measure.