Mathematical Machine Learning Theory, M1407.002700
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Homework 7
Due 5pm, Friday, June 21, 2024

Problem 1: *Orthogonality against* $K(X_i, \cdot)$*.* Let $\mathcal{H}$ be an RKHS on a nonempty set $\mathcal{X}$ with RK K. Let $N \in \mathbb{N}$, $X_1, \dots, X_N \in \mathcal{X}$, and

$$\mathcal{S} = \operatorname{span}\left(\{K(X_i, \cdot)\}_{i=1}^N\right) \subseteq \mathcal{H}.$$

Show if $f \in \mathcal{S}^\perp$, then $f(X_i) = 0$ for all $i = 1, \dots, N$.

Problem 2: *Non-strict representer theorem.* Let $\mathcal{X}$ be a nonempty set, $K\colon \mathcal{X} \times \mathcal{X} \to \mathbf{R}$ a PDK, $\mathcal{H}$ the corresponding RKHS, $X_1, \dots, X_N \in \mathcal{X}$, and $Y_1, \dots, Y_N \in \mathbf{R}$. Consider the optimization problem

$$\underset{f \in \mathcal{H}}{\text{minimize}} \quad L(\{(X_i, Y_i, f(X_i))\}_{i=1}^N) + Q(\|f\|_{\mathcal{H}})$$

where $Q\colon \mathbf{R}_+ \to \mathbf{R}$ is a non-decreasing function. Show that, if a minimizer exists, there is a minimizer in

$$\operatorname{span}\left(\{K(X_i, \cdot)\}_{i=1}^N\right).$$

Problem 3: *Product PDK of of strict PDKs.* Let $\mathcal{X}$ be a nonempty set. Let K_1 and K_2 be strictly PDKs mapping $\mathcal{X} \times \mathcal{X}$ to $\mathbf{R}$. Show that $K_1 K_2$ is strictly a PDK.

Problem 4: *A discontinuous kernel.* Let $\mathcal{X} = \mathbf{R}$. Show that $K\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as

$$K(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

is a positive definite kernel. Furthermore, show that

$$\mathcal{H} = \left\{ f = \sum_{i=1}^N \alpha_i \mathbf{1}_{\{x_i\}} \,\middle|\, N \in \mathbb{N} \cup \{\infty\},\, \{\alpha_i\}_{i=1}^N \subset \mathbf{R},\, \{x_i\}_{i=1}^N \subset \mathbf{R},\, \|f\|_{\mathcal{H}} < \infty \right\}$$

with inner product

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{x \in \mathbf{R}} f(x) g(x) = \sum_{x : f(x) \neq 0,\, g(x) \neq 0} f(x) g(x), \qquad \forall\, f, g \in \mathcal{H}$$

is the corresponding RKHS.