Mathematical and Numerical Optimization, 3341.454
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Additional Problems for Homework 3

Comment on problem 4.4. When convex optimization problem is symmetric with respect to an infinite group, the analysis of problem 4.4 can be carried out using the Haar measure.

Problem 1: *'Hello World' in CVXPY.* Use CVXPY to verify the optimal values you obtained (analytically) for exercise 4.1 in *Convex Optimization.*

Problem 2: *Reformulating constraints in CVXPY.* Each of the following CVXPY code fragments describes a convex constraint on the scalar variables `x`, `y`, and `z`, but violates the CVXPY rule set, and so is invalid. Briefly explain why each fragment is invalid. Then, rewrite each one in an equivalent form that conforms to the CVXPY rule set. In your reformulations, you can use linear equality and inequality constraints, and inequalities constructed using CVXPY functions. You can also introduce additional variables, or use LMIs. Be sure to explain (briefly) why your reformulation is equivalent to the original constraint, if it is not obvious.

Check your reformulations by creating a small problem that includes these constraints, and solving it using CVXPY. Your test problem doesn't have to be feasible; it's enough to verify that CVXPY processes your constraints without error.

Remark. This *looks* like a problem about 'how to use CVXPY software', or 'tricks for using CVXPY'. But it really checks whether you understand the various composition rules, convex analysis, and constraint reformulation rules.

1. `norm([x + 2*y, x - y]) == 0`
2. `square(square(x + y)) <= x - y`
3. `1/x + 1/y <= 1; x >= 0; y >= 0`
4. `norm([max(x,1), max(y,2)]) <= 3*x + y`
5. `x*y >= 1; x >= 0; y >= 0`
6. `(x + y)^2/sqrt(y) <= x - y + 5`
7. `x^3 + y^3 <= 1; x >= 0; y >= 0`
8. `x + z <= 1 + sqrt(x*y - z^2); x >= 0; y >= 0`

Problem 3: *Optimal vehicle speed scheduling.* A vehicle (say, an airplane) travels along a fixed path of n segments, between $n+1$ waypoints labeled $0, \dots, n$. Segment i starts at waypoint $i-1$ and terminates at waypoint i. The vehicle starts at time $t = 0$ at waypoint 0. It travels over each segment at a constant (nonnegative) speed; s_i is the speed on segment i. We have lower and upper limits on the speeds: $s^{\min} \preceq s \preceq s^{\max}$. The vehicle does not stop at the waypoints; it simply proceeds to the next segment. The travel distance of segment i is d_i (which is positive), so the travel time over segment i is d_i/s_i. We let τ_i, $i = 1, \dots, n$, denote the time at which the vehicle arrives at waypoint i. The vehicle is required to arrive at waypoint i, for $i = 1, \dots, n$, between times $\tau_i^{\min}$ and $\tau_i^{\max}$, which are given. The vehicle consumes fuel over segment i at a rate that depends on its speed, $\Phi(s_i)$, where Φ is positive, increasing, and convex, and has units of kg/s.

You are given the data d (segment travel distances), $s^{\min}$ and $s^{\max}$ (speed bounds), $\tau^{\min}$ and $\tau^{\max}$ (waypoint arrival time bounds), and the fuel use function $\Phi : \mathbb{R} \to \mathbb{R}$. You are to choose the speeds $s_1, \ldots, s_n$ so as to minimize the total fuel consumed in kg.

1. Show how to pose this as a convex optimization problem. If you introduce new variables, or change variables, you must explain how to recover the optimal speeds from the solution of your problem. If convexity of the objective or any constraint function in your formulation is not obvious, explain why it is convex.

2. Carry out the method of part (a) on the problem instance with data in `veh_speed_sched_data.py`. Use the fuel use function $\Phi(s_i) = as_i^2 + bs_i + c$ (the parameters a, b, and c are defined in the data file). What is the optimal fuel consumption? Plot the optimal speed versus segment, using the matlab command `stairs` or the function `step` from matplotlib in Python and Julia to better show constant speed over the segments.